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Research article

A combined approach to Perrin and Padovan hybrid sequences

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ABSTRACT

Recently, there has been huge interest to a new numeric set, which brings together three numerical systems: complex, hyperbolic and dual numbers, called as hybrid number. Motivated and based on the importance of this system, we apply the definition of hybrid numbers to Perrin numbers and introduce Perrin hybrid numbers and examine some properties of them. In other words, we obtain their some properties such as Binet-like-formula, partial sums, generating functions, etc. Also we investigate some similar properties with Padovan hybrid sequence. Finally, we give a source Maple 13 code.

1. Introduction

Hybrid numbers are systems of numbers that consists of all three number systems (complex, dual and hyperbolic) together. Since they have this property, they have numerous applications in many areas such as linear algebraic, kinematic, physical applications, etc. Moreover, especially in the last years, a lot of researchers deal with the geometric and physical applications of complex, hyperbolic and dual numbers which are well known two dimensional number systems. For example, Ozdemir [1] introduced the hybrid numbers as a generalization of complex, hyperbolic and dual numbers. The hyperbolic and dual numbers have the same structure with complex numbers. But the hyperbolic unit satisfies $h^2 = 1$, and dual unit satisfies $\varepsilon^2 = 0$. The hybrid number is defined as

$$Z = a + bi + c\varepsilon + dh,$$

where $a, b, c \in \mathbb{R}$ and i, ε, h are operators such that

$$i^2 = -1 \ , \ \varepsilon^2 = 0, \ h^2 = 1 \ ih = -hi = \varepsilon + i.$$

At this content, the author has also given some important properties such as basic operations, norm and classification, representations (matrix and polar form) of hybrid numbers, etc. The author exploits the (Table 1) for the basic multiplication operation:

The conjugate of hybrid number Z is defined as below:

$$\overline{Z} = \overline{a + bi + c\varepsilon + dh} = a - bi - c\varepsilon - dh.$$

The real number $C(Z) = Z\overline{Z} = \overline{Z}Z = a^2 + b^2 - 2bc - d^2$ is called the character of the hybrid number Z.

There are many studies in the literature that concern the special number sequences such as Fibonacci, Lucas, Pell, Jacobsthal, Padovan, Perrin, and etc. The authors, in [2, 3, 4, 5, 6, 7, 8, 9, 10], get some spectacular properties for some well-known number sequences such as Lucas, Pell, Fibonacci, Padovan etc. In [3], $k-$ Fibonacci and $k-$ Lucas numbers are derived and proved by using some special matrices. In [6], some properties of $(k,h)-$ Pell sequence and $(k,h)-$ Pell-Lucas sequence are examined. Furthermore, some formulas for *nth* term and sum of the first n terms of these sequences are calculated. In [7], special circulant matrices with $(k,h)-$ Jacobsthal and Jacobsthal-like sequence are represented. Also, eigenvalues and determinants of circulant matrices involving these sequences are obtained. In [8], matrices formula is given for Padovan and Perrin sequences. In [9], by taking into account matrix properties of new matrix sequences, some behaviours of Padovan and Perrin numbers are investigated. Moreover, some important correspondences between Padovan and Perrin matrix sequences are denoted. In [10], many significant properties of Hessenberg matrices and Pell, Perrin numbers are studied.

Recently, the mentioned number sequences have been combined with hybrid numbers. The authors, in [11, 12, 13, 14, 15, 16, 17, 18], defined some special types of the hybrid numbers and obtain their some spectacular properties.

The Padovan sequence, is the sequence of integers denoted by P_n, is defined by the recursion relation, for all $n \geq 3$:

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Table 1. The hybrid numbers multiplication.

•	i	ε	h
i	-1	$1-h$	$\varepsilon+i$
ε	$h+1$	0	$-\varepsilon$
h	$-\varepsilon+1$	ε	1

$$P_n = P_{n-2} + P_{n-3}$$

with initial values $P_0 = P_1 = P_2 = 1$. The first values of P_n are:

1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, ...

The Perrin sequence, denoted by R_n, is defined by the recursion relation, with initial values $R_0 = 3$, $R_1 = 0$, $R_2 = 2$, for all $n \geq 3$,

$$R_n = R_{n-2} + R_{n-3}.$$

The first values of R_n are given below:

3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39, 51, 68, 90, 139, ...

From [2, 5, 8] and [9], we can give some well-known identities for Padovan sequence (P_n) and Perrin sequence(R_n), as below:

$$P_{n+5} = P_{n+4} + P_n,$$

$$P_n = 2P_{n-2} - P_{n-4} + P_{n-6},$$

$$R_{n+5} = R_{n+4} + R_n,$$

$$R_n = 3P_{n-5} + 2P_{n-4},$$

$$P_{n+m} = P_{n-1}P_{m-1} + P_nP_{m-2} + P_{n-2}P_{m-3},$$

$$R_{n+m} = P_{n-1}R_{m-1} + P_nR_{m-2} + P_{n-2}R_{m-3}.$$

Also, in [2], the authors give the Binet-like-formula for Padovan and Perrin sequences, respectively, as follows:

$$P_n = \frac{(r_2-1)(r_3-1)}{(r_1-r_2)(r_1-r_3)}{r_1}^n + \frac{(r_1-1)(r_3-1)}{(r_2-r_1)(r_2-r_3)}{r_2}^n + \frac{(r_1-1)(r_2-1)}{(r_1-r_3)(r_2-r_3)}{r_3}^n,$$

$$R_n = {r_1}^n + {r_2}^n + {r_3}^n,$$

where r_1, r_2, r_3 are the roots of equation $x^3 - x - 1 = 0$ and

$$r_1 + r_2 + r_3 = 0, \quad r_1r_2r_3 = 1, \quad r_1r_2 + r_2r_3 + r_1r_3 = -1.$$

The Padovan hybrid sequence [14], denoted by HP_n, is defined as below:

$$HP_n = P_n + P_{n+1}i + P_{n+2}\varepsilon + P_{n+3}h, \tag{1}$$

where P_n is the Padovan sequence.

In the light of the above-cited recent works, we define Perrin hybrid numbers. Then, we obtain Binet-like formulas and also represent the partial sums, generating functions and some properties of this sequence.

2. The Perrin hybrid sequences

In this section, we define a new type of hybrid number called the Perrin hybrid numbers and get some interesting properties.

Definition 1. Let us define the Perrin hybrid sequence, denoted by $R_n^{(H)}$, by the relation

$$R_n^{(H)} = R_n + R_{n+1}i + R_{n+2}\varepsilon + R_{n+3}h, \tag{2}$$

where (R_n) is the Perrin sequence, n is a positive integer, $a, b, c \in \mathbb{R}$ and i, ε, h are operators such that $i^2 = -1$, $\varepsilon^2 = 0$, $h^2 = 1$ $ih = -hi = \varepsilon + i$.

From the above definition, the initial values of the Perrin hybrid sequence are given as follows:

$$R_0^{(H)} = 3 + 2\varepsilon + 3h,$$
$$R_1^{(H)} = 2i + 3\varepsilon + 2h,$$
$$R_2^{(H)} = 2 + 3i + 2\varepsilon + 5h.$$

Theorem 1. Let $n \geq 0$ be an integer. Then the character of the Perrin hybrid number is:

$$C\left(R_n^H\right) = -2R_{n+1}(R_{n+2} + R_n).$$

Proof. From the definition of the Perrin sequence, we have

$$C\left(R_n^H\right) = {R_n}^2 + {R_{n+1}}^2 - 2R_{n+1}R_{n+2} - {R_{n+3}}^2.$$

Then we get

$$\begin{aligned} C\left(R_n^H\right) &= R_n^2 + R_{n+1}^2 - 2R_{n+1}R_{n+2} - R_{n+3}^2 \\ &= {R_n}^2 + {R_{n+1}}^2 - 2R_{n+1}R_{n+2} - {R_n}^2 - {R_{n+1}}^2 - 2R_nR_{n+1} \\ &= -2R_{n+1}(R_{n+2} + R_n). \end{aligned}$$

Hence, the proof is completed.

Corollary 1. The norm of the Perrin hybrid sequence $R_n^{(H)}$ is given by

$$\left\|R_n^{(H)}\right\|^2 = 2R_{n+1}(R_{n+2} + R_{n+1}).$$

Proof. By using the definition of character and norm for hybrid numbers, we have

$$\left\|R_n^{(H)}\right\| = \sqrt{\left|C\left(R_n^{(H)}\right)\right|} = \sqrt{\left|-2R_{n+1}(R_{n+2} + R_{n+1}\right|)}.$$

Therefore, we have

$$\left\|R_n^{(H)}\right\|^2 = 2R_{n+1}(R_{n+2} + R_{n+1}).$$

Theorem 2. Let $n \geq 0$ be an integer. Then

$$R_n^{(H)} = \sum_{k=1}^{4}\left(1 + ir_k + \varepsilon r_k^2 + hr_k^3\right){r_k}^n.$$

Proof. From the Binet-like-formula of the Perrin sequence, we have

$$R_n = {r_1}^n + {r_2}^n + {r_3}^n,$$

where r_1, r_2, r_3 are the roots of equation $x^3 - x - 1 = 0$.

Exploiting the definition of the Perrin hybrid sequence $R_n^{(H)}$, the proof can be seen easily.

Lemma 1. Let $n \geq 0$ be an integer. Then

$$\sum_{k=0}^{n} R_k = R_{n+5} - 2.$$

Proof. For details, see [4].

Theorem 3. Let $n \geq 0$ be an integer. Then

$$\sum_{k=0}^{n} R_k^{(H)} = R_{n+5}^{(H)} - (2 + 3i + 3\varepsilon + 5h).$$

Proof. It is known that $\sum_{k=0}^{n} R_k^{(H)} = R_0^{(H)} + R_1^{(H)} + R_2^{(H)} + \cdots + R_n^{(H)}$. Thus, we obtain

$$\begin{aligned}&\sum_{k=0}^{n}R_k^{(H)}=(R_0+R_1i+R_2\varepsilon+R_3h)+(R_1+R_2i+R_3\varepsilon+R_4h)+...\\&+(R_n+R_{n+1}i+R_{n+2}\varepsilon+R_{n+3}h)\\&=(R_0+R_1+R_2+...+R_n)+(R_1+R_2+...+R_{n+1}+R_0-R_0)i\\&+(R_2+R_3+...+R_{n+2}+R_0+R_1-R_0-R_1)\varepsilon+(R_3+R_4+...\\&+R_{n+3}+R_0+R_1+R_2-R_0-R_1-R_2)h\\&=\sum_{k=0}^{n}R_k+\left(\sum_{k=0}^{n+1}R_k-3\right)i+\left(\sum_{k=0}^{n+2}R_k-3\right)\varepsilon+\left(\sum_{k=0}^{n+3}R_k-5\right)h.\end{aligned}$$

Therefore, we get

$$\begin{aligned}\sum_{k=0}^{n}R_k^{(H)}&=R_{n+5}-2+(R_{n+6}-2-1)i+(R_{n+7}-2-1)\varepsilon+(R_{n+8}-2-3)h\\&=R_{n+5}+R_{n+6}i+R_{n+7}\varepsilon+R_{n+8}h-(2+3i+4\varepsilon+5h)\\&=R_{n+5}^{(H)}-(2+3i+4\varepsilon+5h).\end{aligned}$$

Thus, the proof is completed.

By considering some properties of Padovan hybrid sequence and Perrin hybrid sequence, let us give the following theorem:

Theorem 4. Let $n\geq 0$ be an integer. Then

a) $$\begin{aligned}HP_{n+1}+HP_n&=\alpha r_1^n(1+r_1i+r_1^2\varepsilon+r_1^3h)(1+r_1)\\&=\beta r_2^n(1+r_2i+r_2^2\varepsilon+r_2^3h)(1+r_2)\\&=\gamma r_3^n(1+r_3i+r_3^2\varepsilon+r_3^3h)(1+r_3),\end{aligned}$$

b) $$\begin{aligned}R_{n+1}^{(H)}+R_n^{(H)}&=r_1^n(1+r_1i+r_1^2\varepsilon+r_1^3h)(1+r_1)\\&=r_2^n(1+r_2i+r_2^2\varepsilon+r_2^3h)(1+r_2)\\&=r_3^n(1+r_3i+r_3^2\varepsilon+r_3^3h)(1+r_3),\end{aligned}$$

c) $$\begin{aligned}HP_{n+1}-HP_n&=\alpha r_1^n(1+r_1i+r_1^2\varepsilon+r_1^3h)(1-r_1)\\&=\beta r_2^n(1+r_2i+r_2^2\varepsilon+r_2^3h)(1-r_2)\\&=\gamma r_3^n(1+r_3i+r_3^2\varepsilon+r_3^3h)(1-r_3),\end{aligned}$$

d) $$\begin{aligned}R_{n+1}^{(H)}-R_n^{(H)}&=r_1^n(1+r_1i+r_1^2\varepsilon+r_1^3h)(1-r_1)\\&=r_2^n(1+r_2i+r_2^2\varepsilon+r_2^3h)(1-r_2)\\&=r_3^n(1+r_3i+r_3^2\varepsilon+r_3^3h)(1-r_3).\end{aligned}$$

where

$$\alpha=\frac{(r_2-1)(r_3-1)}{(r_1-r_2)(r_1-r_3)},\beta=\frac{(r_1-1)(r_3-1)}{(r_2-r_1)(r_2-r_3)},\gamma=\frac{(r_1-1)(r_2-1)}{(r_1-r_3)(r_2-r_3)}.$$

Proof. By exploiting the Theorem 2, we have

$$\begin{aligned}HP_n=\alpha r_1^n(1+r_1i+r_1^2\varepsilon+r_1^3h)+\beta r_2^n(1+r_2i+r_2^2\varepsilon+r_2^3h)+\gamma r_3^n(1+r_3i\\+r_3^2\varepsilon+r_3^3h),\end{aligned}$$

where

$$\alpha=\frac{(r_2-1)(r_3-1)}{(r_1-r_2)(r_1-r_3)},\beta=\frac{(r_1-1)(r_3-1)}{(r_2-r_1)(r_2-r_3)},\gamma=\frac{(r_1-1)(r_2-1)}{(r_1-r_3)(r_2-r_3)}.$$

Thus, we write

$$\begin{aligned}HP_{n+1}+HP_n&=\alpha r_1^{n+1}(1+r_1i+r_1^2\varepsilon+r_1^3h)\\&+\beta r_2^{n+1}(1+r_2i+r_2^2\varepsilon+r_2^3h)\\&+\gamma r_3^{n+1}(1+r_3i+r_3^2\varepsilon+r_3^3h)\\&+\alpha r_1^n(1+r_1i+r_1^2\varepsilon+r_1^3h)\\&+\beta r_2^n(1+r_2i+r_2^2\varepsilon+r_2^3h)\\&+\gamma r_3^n(1+r_3i+r_3^2\varepsilon+r_3^3h).\end{aligned}$$

According to the common factors of the right side of the equality, we get

$$\begin{aligned}HP_{n+1}+HP_n&=\alpha r_1^{n+1}(1+r_1i+r_1^2\varepsilon+r_1^3h)(1+r_1)\\&+\beta r_2^n(1+r_2i+r_2^2\varepsilon+r_2^3h)(1+r_2)\\&+\gamma r_3^n(1+r_3i+r_3^2\varepsilon+r_3^3h)(1+r_3),\end{aligned}$$

which is desired. So the proof is completed. The others can be proved by similar manner.

In [1], the complex, hyperbolic and dual units are given as

$$i\to\begin{bmatrix}0&1\\-1&0\end{bmatrix},\ \varepsilon\to\begin{bmatrix}1&-1\\1&-1\end{bmatrix},\ h\to\begin{bmatrix}0&1\\1&0\end{bmatrix}.$$

For matrix representation of Perrin hybrid numbers, with $n\in N$, is obtained as:

$$R_n^{(H)}=R_n\begin{bmatrix}1&0\\0&1\end{bmatrix}+R_{n+1}\begin{bmatrix}0&1\\-1&0\end{bmatrix}+R_{n+2}\begin{bmatrix}1&-1\\1&-1\end{bmatrix}+R_{n+3}\begin{bmatrix}0&1\\1&0\end{bmatrix}.$$

Thus, we have

$$R_n^{(H)}=\begin{bmatrix}R_n+R_{n+2}&R_{n+1}-R_{n+2}+R_{n+3}\\-R_{n+1}+R_n&R_n-R_{n+2}\end{bmatrix}.$$

Let us take $\alpha_0,\alpha_1,\alpha_2,...$ sequence, where $\alpha_0,\alpha_1,..$ are real numbers. Then, the function

$$h(x)=\alpha_0+\alpha_1x+\alpha_2x^2+\ldots+\alpha_nx^n+\ldots$$

is called the generating function for the sequence α_n. If we want to write the generating function for finite sequence α_0, α_1, α_2, ..., α_n where $\alpha_i=0$ for $i>n$, that is;

$$h(x)=\alpha_0+\alpha_1x+\alpha_2x^2+...+\alpha_nx^n.$$

Here, we will use generating functions as above:

$$\frac{1}{1-\alpha x}=1\ +\alpha x+\alpha^2x^2+...+\alpha^nx^n=\sum_{n=0}^{\infty}\alpha^nx^n,$$

where α_n is real number for each $n\in\mathbb{N}$.

Theorem 5. The generating function for Perrin hybrid sequence $R_n^{(H)}$ is

$$\sum_{n=0}^{\infty}R_n^{(H)}x^n=\frac{R_0^{(H)}+R_1^{(H)}x+(-1+3i+2h)x^2}{1-x^2-x^3}.$$

Proof. Suppose that the generating function of the Perrin hybrid sequence $R_n^{(H)}$ has the form

$$g(x)=\sum_{n=0}^{\infty}R_n^{(H)}x^n=R_0^{(H)}+R_1^{(H)}x+R_2^{(H)}x^2+R_3^{(H)}x^3+\cdots\ .$$

Then we have

$$x^2g(x)=R_0^{(H)}x^2+R_1^{(H)}x^3+R_2^{(H)}x^4+R_3^{(H)}x^5+\cdots\ ,$$

and

$$x^3g(x)=R_0^{(H)}x^3+R_1^{(H)}x^4+R_2^{(H)}x^5+R_3^{(H)}x^6+\cdots\ .$$

Thus, we obtain

$$\begin{aligned}&g(x)-x^2g(x)-x^3g(x)=\left(R_0^{(H)}+R_1^{(H)}x+R_2^{(H)}x^2+R_3^{(H)}x^3+\cdots\right)\\&-\left(R_0^{(H)}x^2+R_1^{(H)}x^3+R_2^{(H)}x^4+R_3^{(H)}x^5+\cdots\right)\\&-\left(R_0^{(H)}x^3+R_1^{(H)}x^4+R_2^{(H)}x^5+R_3^{(H)}x^6+\cdots\right)\\&=\left(R_0^{(H)}+R_1^{(H)}x\right)+\left(R_2^{(H)}-R_0^{(H)}\right)x^2+\left(R_3^{(H)}-R_1^{(H)}-R_0^{(H)}\right)x^2+\cdots\\&+\left(R_n^{(H)}-R_{n-2}^{(H)}-R_{n-3}^{(H)}\right)x^n+\cdots\end{aligned}$$

We deduce that

$$g(x)-x^2g(x)-x^3g(x)=R_0^{(H)}+R_1^{(H)}x+\left(R_2^{(H)}-R_0^{(H)}\right)x^2.$$

Thus, we get

$$g(x)\left(1-x^2-x^3\right)=R_0^{(H)}+R_1^{(H)}x+(i+\varepsilon+h)x^2.$$

Therefore, we have

$$\sum_{n=0}^{\infty}R_n^{(H)}x^n=\frac{R_0^{(H)}+R_1^{(H)}x+(i+\varepsilon+h)x^2}{1-x^2-x^3}.$$

Taking into account the Padovan numbers, Morales [5] has defined the following sequences:

$$Q_n=P_n+P_{-n}$$

and

$$\acute{Q}_n=P_n-P_{-n},$$

where $n\geq 0$ is an integer and $P_{-n}=-P_{-(n-1)}+P_{-(n-3)}$.

Additionally, the author gives the following table:

n	1	2	3	4	5	6	7	8	9	10	11	12
P_n	1	1	2	2	3	4	5	7	9	12	16	21
P_{-n}	0	1	0	0	1	-1	1	0	-1	2	-2	1

Here, by following the same method, let us consider the Perrin numbers as below:

$$R_{-n}=-R_{-(n-1)}+R_{-(n-3)},$$

where R_n the *nth* Perrin number:

n	1	2	3	4	5	6	7	8	9	10	11	12
R_n	0	2	3	2	5	5	7	10	12	17	22	39
R_{-n}	-1	1	2	-3	4	-2	-1	5	-7	6	-1	-6

Now we define the new sequences $Q_n^{(H)}$, $Q_n^{'(H)}$, $T_n^{(H)}$ and $T_n^{'(H)}$.

Definition 2. Let $n\geq 0$ be an integer. Let us define

$$Q_n^{(H)}=P_n^{(H)}+P_{-n}^{(H)},\quad Q_n^{'(H)}=P_n^{(H)}-P_{-n}^{(H)}$$

and

$$T_n^{(H)}=R_n^{(H)}+R_{-n}^{(H)},\quad T_n^{'(H)}=R_n^{(H)}-R_{-n}^{(H)},$$

where

$$P_{-n}^{(H)}=P_{-n}+P_{-(n+1)}i+P_{-(n+2)}\varepsilon+P_{-(n+3)}h$$

and

$$R_{-n}^{(H)}=R_{-n}+R_{-(n+1)}i+R_{-(n+2)}\varepsilon+R_{-(n+3)}h.$$

According to these new sequences, we can give the following table for the first terms of sequences $P_n^{(H)}$, $P_{-n}^{(H)}$, $Q_n^{(H)}$ and $Q_n^{'(H)}$:

n	1	2	3	4	5
$P_n^{(H)}$	$1+i+2\varepsilon+2h$	$1+2i+2\varepsilon+3h$	$2+2i+3\varepsilon+4h$	$2+3i+4\varepsilon+5h$	$3+4i+5\varepsilon+7h$
$P_{-n}^{(H)}$	i	$1+h$	$\varepsilon-h$	$i-\varepsilon+h$	$1-i+\varepsilon$
$Q_n^{(H)}$	$1+2i+2\varepsilon+2h$	$2+2i+2\varepsilon+4h$	$2+2i+4\varepsilon+3h$	$2+4i+3\varepsilon+6h$	$4+3i+6\varepsilon+7h$
$Q_n^{'(H)}$	$1+2\varepsilon+2h$	$2i+2\varepsilon+2h$	$2+2i+2\varepsilon+5h$	$2+2i+5\varepsilon+4h$	$2+5i+4\varepsilon+7h$

As an example, for $n=2$, we have

$$P_{-2}^{(H)}=P_{-2}+P_{-3}i+P_{-4}\varepsilon+P_{-5}h=1+0\mathrm{i}+0\varepsilon+\mathrm{h}=1+\mathrm{h}.$$

Additionally, one can find the following table for the first terms of sequences $R_n^{(H)}$, $R_{-n}^{(H)}$, $T_n^{(H)}$ and $T_n^{'(H)}$:

n	1	2	3	4	5
$R_n^{(H)}$	$2i+3\varepsilon+2h$	$2+3i+2\varepsilon+5h$	$3+2i+5\varepsilon+5h$	$2+5i+5\varepsilon+7h$	$5+5i+7\varepsilon+10h$
$R_{-n}^{(H)}$	$-1+i+2\varepsilon-3h$	$1+2i-3\varepsilon+4h$	$2-3i-4\varepsilon-2h$	$-3+4i-2\varepsilon-h$	$4-2i-\varepsilon+5h$
$T_n^{(H)}$	$-1+3i+5\varepsilon-h$	$3+5i-\varepsilon+9h$	$5-i+\varepsilon+3h$	$-1+9i+3\varepsilon+6h$	$9+3i+6\varepsilon+15h$
$T_n^{'(H)}$	$1+i+\varepsilon+5h$	$2+i+5\varepsilon+h$	$1+5i+9\varepsilon+7h$	$5-i+7\varepsilon+8h$	$1+7i+8\varepsilon+5h$

For $n=1$, we have

$$R_{-1}^{(H)}=R_{-1}+R_{-2}i+R_{-3}\varepsilon+R_{-4}h=-1+\mathrm{i}+2\ \varepsilon-3\mathrm{h}.$$

Theorem 6. Let $n\geq 0$ be an integer. Then

(1) $HP_{n+1}=Q_n^{(H)}+Q_{n-1}^{(H)}-Q_{n-3}^{(H)}$,

(2) $R_{n+1}^{(H)}=T_n^{(H)}+T_{n-1}^{(H)}-T_{n-3}^{(H)}$.

Proof. Let $n\geq 0$.be an integer. By definition of $Q_n^{(H)}$, we have

$$Q_n^{(H)}=HP_n+HP_{-n},Q_{n-1}^{(H)}=HP_{n-1}+HP_{-(n-1)}$$

and

$$Q_{n-3}^{(H)}=HP_{n-3}+HP_{-(n-3)}.$$

Thus, we get

$$\begin{aligned}
Q_n^{(H)}+Q_{n-1}^{(H)}-Q_{n-3}^{(H)}&=\Big[(P_n+P_{n+1}i+P_{n+2}\varepsilon+P_{n+3}h)\\
&+\left(P_{-n}+P_{-(n+1)}i+P_{-(n+2)}\varepsilon+P_{-(n+3)}h\right)\Big]\\
&+\Big[\left(P_{n-1}+P_ni+P_{n+1}\varepsilon+P_{n+2}h\right)+\left(P_{-(n-1)}+P_{-n}i+P_{-(n+1)}\varepsilon+P_{-(n+2)}h\right)\Big]\\
&-\Big[\left(P_{n-3}+P_{n-2}i+P_{n-1}\varepsilon+P_nh\right)\Big]\\
&=\Big[\left(P_n+P_{-n}\right)+\left(P_{n-1}+P_{-(n-1)}\right)-\left(P_{n-3}+P_{-(n-3)}\right)\Big]\\
&+\Big[\left(P_n+P_{-n}\right)+\left(P_{n-1}+P_{-(n-1)}\right)-\left(P_{n-3}+P_{-(n-3)}\right)\Big]i\\
&+\Big[\left(P_n+P_{-n}\right)+\left(P_{n-1}+P_{-(n-1)}\right)-\left(P_{n-3}+P_{-(n-3)}\right)\Big]\varepsilon\\
&+\Big[\left(P_n+P_{-n}\right)+\left(P_{n-1}+P_{-(n-1)}\right)-\left(P_{n-3}+P_{-(n-3)}\right)\Big]h.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
Q_n^{(H)}+Q_{n-1}^{(H)}-Q_{n-3}^{(H)}&=(Q_n+Q_{n-1}-Q_{n-3})+(Q_{n+1}+Q_n-Q_{n-2})i\\
&+(Q_{n+2}+Q_{n+1}-Q_{n-1})\varepsilon+(Q_{n+3}+Q_{n+2}-Q_n)h.
\end{aligned}$$

Then, we get

$$Q_n^{(H)}+Q_{n-1}^{(H)}-Q_{n-3}^{(H)}=P_{n+1}+P_{n+2}i+P_{n+3}\varepsilon+P_{n+4}h=P_{n+1}^{(H)}.$$

Thus, the proof is completed.

By similar way, one can prove that

$$R_{n+1}^{(H)}=T_n^{(H)}+T_{n-1}^{(H)}-T_{n-3}^{(H)}.$$

3. Conclusion

Recently, a new numeric set was defined, by Özdemir in [1], called as hybrid number, which brings together three numerical systems: complex, hyperbolic and dual numbers. Motivated by the work [1] and based on the importance of this system, we apply the definition of hybrid numbers to Perrin numbers and introduce Perrin hybrid numbers and examine some properties of them. Additionally, we give a Maple 13 source code to get them.

```
restart:
with(LinearAlgebra):with(linalg):
n:= -:
R:= proc(n)
if n = 0 then
return 3:
elif n = 1 then
return 0:
elif n = 2 then
return 2:
elif n = 3 then
return 3:
else
return R(n−2)+R(n−3)
end if;
end proc;
R(n):
R: = proc(n)
if n = 0 then return 3
elif n = 1 then return 0
elif n = 2 then return 2
elif n = 3 then return 3
else return R(n−2) + R(n−3)
end if
end proc
```

$R_n^{(H)} := R_n + R_{n+1}i + R_{n+2}\varepsilon + R_{n+3}h :$

Declarations

Author contribution statement

S. H. Jafari Petroudi, M. Pirouz, E. Karaca, F. Yilmaz: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

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Data will be made available on request.

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The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

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