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On Harmonic Complex Balancing Numbers

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Abstract: In the present work, we define harmonic complex balancing numbers by considering well-known balancing numbers and inspiring harmonic numbers. Mainly, we investigate some of their basic characteristic properties such as the Binet formula and Cassini identity, etc. In addition, one type of symmetric matrix family whose entries are harmonic complex balancing numbers is constructed. Additionally, some linear algebraic properties are obtained. Furthermore, some inequalities are stated by exploiting the well-known inequalities between various matrix norms. Finally, we illustrate the results with some numerical examples.

Keywords: balancing number; Binet formula; Cassini identity; matrix norm; complex balancing number

MSC: 15A60;11B37; 11B83; 11B39

1. Introduction

In mathematics, the well-known second-order homogeneous linear recurrence $\{w_n \equiv w_n(a,b;p,q)\}$ is defined by A. F. Horadam, as below:

$$w_n = pw_{n-1} - qw_{n-2}\,, \quad \text{for} \quad n \geq 2,$$

with initial conditions

$$w_0 = a \text{ and } w_1 = b\,,$$

where a and b are arbitrary integers [1]. In the literature, the Lucas sequences $U_n(p,q)$ and $V_n(p,q)$ are certain constant-recursive integer sequences that satisfy Horadam's recurrence relation given above. In other words, any sequence satisfying this recurrence relation can be represented as a linear combination of the Lucas sequences $U_n(p,q)$ and $V_n(p,q)$. The Fibonacci numbers, Mersenne, Pell, Lucas, Jacobsthal, and balancing numbers are considered the famous examples of Lucas sequences. These sequences satisfy many common properties and identities, for example Binet formulas, Catalan identities, and matrix representation. Recently, these sequences were investigated, generalized, or extended; for example, see [2–5]. When $p = 6$ and $q = 1$, the sequence w_n is called the sequence of balancing numbers (please see [6]) and denoted by B_n, that is

$$B_{n+1} = 6B_n - B_{n-1}, \quad B_0 = 1, B_1 = 6. \tag{1}$$

Balancing numbers have several nice properties; for example, $8B_n^2 + 1$ is a perfect square for any integer n. Moreover, from [6], the characteristic equation of balancing numbers is $x^2 - 6x + 1 = 0$. This equation has two real roots: $\alpha = 3 + \sqrt{8}$, $\beta = 3 - \sqrt{8}$. The generating function of the balancing sequence is

$$f(x) = \frac{1}{1 - 6x + x^2}. \tag{2}$$

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Using the generating function, given by (2), the Binet's formula for B_n is obtained as follows:

$$B_n = M\alpha^n + N\beta^n = M\left(3+\sqrt{8}\right)^n + N\left(3-\sqrt{8}\right)^n \tag{3}$$

where

$$M = \frac{\alpha}{\alpha-\beta}, \quad N = \frac{-\beta}{\alpha-\beta}. \tag{4}$$

In the literature, there are many interesting papers that have examined balancing numbers and their spectacular properties. For example, G. K. Panda [7] established many important identities concerning balancing numbers and their related sequences. A. Berczes et al. [8] and P. Olajos [9] surveyed many interesting properties of generalized balancing numbers. Recently, G. K. Panda et al. [10] introduced gap balancing numbers and established many properties of these numbers. Some curious congruence properties of balancing numbers were also studied in [11]. In [12,13], Ray established new product formulae to generate balancing numbers. In [14], the authors associated the balancing numbers with graphs. In [15], the authors study on Gaussian balancing and Gaussian cobalancing quaternions.

The nth harmonic number, denoted by H_n, is defined by

$$H_n = \sum_{k=1}^{n} \frac{1}{k},$$

where $H_0 = 0$. In [16], the nth harmonic number H_n can also be expressed by Stirling numbers as follows:

$$H_n = \frac{S(n+1,2)}{n!}$$

where $S(n,2)$ is the Stirling numbers of the second kind.

From [17–19], we have some interesting properties of harmonic numbers:

$$\sum_{k=1}^{n-1} H_k = nH_n - n,, \tag{5}$$

$$\sum_{k=1}^{n} H_k^2 = (n+1)H_{n+1}^2 - (2n+3)H_{n+1} + 2n + 2, \tag{6}$$

$$\sum_{k=1}^{n} kH_k^2 = \frac{(n-1)(n+2)}{2}H_{n+1}^2 - \frac{n^2-3n-7}{2}H_{n+1} + \frac{n^2-9n-10}{4}. \tag{7}$$

In [20], the difference operator of any $f(x)$ function is defined as

$$\Delta f(x) = f(x+1) - f(x) \tag{8}$$

and the anti-difference operator $\sum$ has the following properties:

$$\sum_{a}^{b} u(x)\Delta v(x)\delta_x = u(x)v(x)|_a^{b+1} - \sum_{a}^{b} Ev(k)\Delta u(x)\delta_x, \tag{9}$$

where $Ev(x) = v(x+1)$. In [20], the authors used a property of the finite difference operator to show the validity of the identity (5). In [21,22], the authors investigated harmonic complex Fibonacci sequences and harmonic hybrid Fibonacci numbers.

In [23], the authors considered the following square matrix M:

$$M = \begin{bmatrix} K & b \\ b^T & c \end{bmatrix}$$

where K is an $n \times n$ nonsingular matrix, b is an $n \times 1$ matrix, and c is a real number used to obtain the inverse of M, as follows:

$$M^{-1} = \begin{bmatrix} K^{-1} + \frac{1}{l}K^{-1}bb^TK^{-1} & -\frac{1}{l}K^{-1}b \\ -\frac{1}{l}b^TK^{-1} & \frac{1}{l} \end{bmatrix}, \tag{10}$$

where $l = c - b^TK^{-1}b$.

It is obvious that the norm of a matrix is a non-negative real number. There are several different methods of defining a matrix norm, but they all share the same definite characteristics. Let $A = (a_{ij})$ be an $n \times n$ matrix; then, let us remember some well-known matrix norm types. For example, the maximum column length norm, denoted by $c_1(.)$, and the maximum row length norm, denoted by $r_1(.)$, are defined as follows:

$$c_1(A) = max_j \sqrt{\sum_i |a_{ij}|^2} \quad , \quad r_1(A) = max_i \sqrt{\sum_j |a_{ij}|^2}. \tag{11}$$

The ℓ_p norm of A is defined by

$$\|A\|_p = (\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^p)^{\frac{1}{p}}. \tag{12}$$

For $p = 2$, this norm is called Frobenius or Euclidean norm and denoted by $\|A\|_{\mathbb{E}}$. The spectral norm of A is defined by

$$\|A\|_2 = \sqrt{\max_{1 \le i \le n} \lambda_i}, \tag{13}$$

where λ_i is the eigenvalue of matrix AA^H; here, A^H is a conjugate transpose of matrix A.

Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ real matrices; then, the Hadamard product of these matrices (please see [24]) is defined as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \circ \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} & a_{13}b_{13} \\ a_{21}b_{21} & a_{22}b_{22} & a_{23}b_{23} \\ a_{31}b_{31} & a_{32}b_{32} & a_{33}b_{33} \end{bmatrix}.$$

For $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, the Hadamard product is defined by $A \circ B = (a_{ij}b_{ij})$, and if $A = B \circ C$, the Hadamard product of B and C satisfies the following:

$$\|A\|_2 \le r_1(B)c_1(C). \tag{14}$$

There is a relation between the Frobenius and spectral norms, that is

$$\frac{1}{\sqrt{n}}\|A\|_{\mathbb{E}} \le \|A\|_2 \le \|A\|_{\mathbb{E}}. \tag{15}$$

In [25], the authors defined a particular $n \times n$ matrix $H = [H_{k_{i,j}}]_{i,j=1}^{n}$ and its Hadamard exponential matrix $e^{\circ H} = [e^{H_{k_{i,j}}}]$, where $k_{i,j} = min(i, j)$ and H_n is the n^{th} harmonic number. Then, they investigated the determinants and inverses of these matrices. Moreover, they presented the Euclidean norm and two upper bounds and lower bounds for the spectral norm of these matrices. In [26], the authors defined harmonic Pell numbers. In addition, they constructed one type of symmetric matrix family whose elements are harmonic Pell numbers and its Hadamard exponential matrix. They investigated some linear algebraic

properties and obtained inequalities by using matrix norms. Furthermore, some summation identities for harmonic Pell numbers were obtained.

Recently, a new approach to the significant extension of famous number sequences into the complex plane has arisen. For example, in [27], the authors defined complex Fibonacci sequence F_n^* as follows:

$$F_n^* = F_n + iF_{n+1}, \quad F_0^* = i, F_1^* = 1 + i. \tag{16}$$

Additionally, in [28], complex Lucas numbers were defined by the following recurrence relation $L_n^* = L_n + iL_{n+1}$. Note that researchers can find more papers about different kinds of complex number sequences in the literature.

In this paper, we give a new approach to generalizing balancing numbers. In this content, we initially give complex balancing sequences, denoted by $\mathbb{B}_n$, and then define harmonic complex balancing sequences, denoted by $\mathcal{H}B_n$. Then, we obtain some properties such as the Binet formula, Cassini identity, etc. Furthermore, taking into account one type of symmetric matrix family, we obtain some linear algebraic properties and obtain interesting inequalities.

2. The Harmonic Complex Balancing Numbers

In this section, we initially define complex balancing and harmonic complex balancing numbers. Then, we present some excellent results involving them.

Definition 1. *Let us define complex balancing numbers with the following recurrence relation:*

$$\mathbb{B}_n = B_n + iB_{n+1} \tag{17}$$

where $\mathbb{B}_0 = 1 + 6i, \mathbb{B}_1 = 6 + 35i$.

We give the first few terms of the complex balancing numbers in Table 1.

Table 1. Some Complex Balancing Numbers.

n	0	1	2	3	4
$\mathbb{B}_n$	$1 + 6i$	$6 + 35i$	$35 + 204i$	$204 + 1189i$	$1189 + 6930i$

Theorem 1. *(Generating function) The generating function for the complex balancing numbers is given by*

$$g(x) = \sum_{n=0}^{\infty} (\mathbb{B}_n)x^n = \frac{1 + i(6 - x)}{1 - 6x + x^2}.$$

Proof. By exploiting the definition of the generating function, we have

$$\begin{aligned} g(x) &= \mathbb{B}_0 + \mathbb{B}_1 x + \mathbb{B}_2 x^2 + \mathbb{B}_3 x^3 + ... + \mathbb{B}_n x^n + ... \\ -6xg(x) &= -6\mathbb{B}_0 x - 6\mathbb{B}_1 x^2 - 6\mathbb{B}_2 x^3 - ... - 6\mathbb{B}_{n-1} x^n - ... \\ x^2 g(x) &= \mathbb{B}_0 x^2 + \mathbb{B}_1 x^3 + \mathbb{B}_2 x^4 + ... + \mathbb{B}_{n-2} x^n + ... \end{aligned}$$

from here,

$$\begin{aligned}(1-6x+x^2)g(x) &= \mathbb{B}_0+(\mathbb{B}_1-6\mathbb{B}_0)x\\ &\quad+(\mathbb{B}_2-6\mathbb{B}_1+\mathbb{B}_0)x^2\\ &\quad+(\mathbb{B}_3-6\mathbb{B}_2+\mathbb{B}_1)x^3\\ &\quad.\\ &\quad.\\ &\quad.\\ &\quad+(\mathbb{B}_n-6\mathbb{B}_{n-1}+\mathbb{B}_{n-2})x^n+...\end{aligned}$$

and as a result;

$$\begin{aligned}g(x) &= \frac{\mathbb{B}_0+(\mathbb{B}_1-6\mathbb{B}_0)x}{1-6x+x^2}\\ &= \frac{(1+6i)+((6+35i)-6(1+6i))x}{1-6x+x^2}.\\ &= \frac{1+i(6-x)}{1-6x+x^2}.\end{aligned}$$

Therefore, the proof is completed. □

Theorem 2. *(Binet formula) For $n \geq 0$,*

$$\mathbb{B}_n = M\alpha^n\alpha_* + N\beta^n\beta_* \tag{18}$$

where

$$M = \frac{\alpha}{\alpha-\beta} \quad and \quad N = \frac{-\beta}{\alpha-\beta} \tag{19}$$

$$\alpha_* = 1+i\alpha \quad and \quad \beta_* = 1+i\beta. \tag{20}$$

Proof. By using the Binet formula for the balancing numbers, it is easy to see the proof. □

Definition 2. *Let us define harmonic complex balancing numbers as follows:*

$$\mathcal{H}B_n = \sum_{k=1}^{n}\frac{1}{B_k+iB_{k+1}}$$

where $\mathcal{H}B_n$ denotes the nth harmonic complex balancing number, B_k denotes the kth balancing number, and $\mathcal{H}B_0 = 0$.

Example 1. *Let us calculate the harmonic complex balancing number for n = 3,*

$$\begin{aligned}\mathcal{H}B_3 &= \sum_{k=1}^{3}\frac{1}{B_k+iB_{k+1}}\\ &= \frac{1}{B_1+iB_2}+\frac{1}{B_2+iB_3}+\frac{1}{B_3+iB_4}\\ &= \frac{1}{6+35i}+\frac{1}{35+204i}+\frac{1}{204+1189i}\\ &= \frac{34564649477-201599438424i}{6047764964449}.\end{aligned}$$

Theorem 3. *(Binet formula) The Binet formula for the nth harmonic complex balancing number is given by*

$$\mathcal{H}B_n = \sum_{k=1}^{n} \frac{1}{(M\alpha^k + N\beta^k) + i(M\alpha^{k+1} + N\beta^{k+1})}$$

where

$$\begin{aligned} M &= \frac{\alpha}{\alpha-\beta} = \frac{3\sqrt{2}+4}{8} \\ N &= -\frac{\beta}{\alpha-\beta} = \frac{4-3\sqrt{2}}{8} \\ \alpha, \beta = 3 \pm \sqrt{8}, \quad \alpha - \beta &= 4\sqrt{2}. \end{aligned}$$

Proof. Let $B_n = \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}$ be as in (3). Then, we obtain

$$\begin{aligned} \mathcal{H}B_n &= \sum_{k=1}^{n} \frac{1}{\left(\frac{\alpha^{k+1}-\beta^{k+1}}{\alpha-\beta}\right) + i\left(\frac{\alpha^{k+2}-\beta^{k+2}}{\alpha-\beta}\right)} \\ &= \sum_{k=1}^{n} \frac{1}{\left(\frac{\alpha^{k+1}}{\alpha-\beta} - \frac{\beta^{k+1}}{\alpha-\beta}\right) + i\left(\frac{\alpha^{k+2}}{\alpha-\beta} - \frac{\beta^{k+2}}{\alpha-\beta}\right)} \\ &= \sum_{k=1}^{n} \frac{1}{\left((\frac{\alpha}{\alpha-\beta})\alpha^k + (-\frac{\beta}{\alpha-\beta})\beta^k\right) + i\left((\frac{\alpha}{\alpha-\beta})\alpha^{k+1} + (-\frac{\beta}{\alpha-\beta})\beta^{k+1}\right)} \end{aligned}$$

where $M = \frac{\alpha}{\alpha-\beta}, N = -\frac{\beta}{\alpha-\beta}$. Hence, we have

$$\sum_{k=1}^{n} \frac{1}{(M\alpha^k + N\beta^k) + i(M\alpha^{k+1} + N\beta^{k+1})}.$$

□

Example 2. *For $n = 2$,*

$$\begin{aligned} \sum_{k=1}^{2} \frac{1}{B_k + iB_{k+1}} &= \frac{1}{(M\alpha + N\beta) + i(M\alpha^2 + N\beta^2)} + \frac{1}{(M\alpha^2 + N\beta^2) + i(M\alpha^3 + N\beta^3)} \\ &= \frac{301181}{54022501} - i\frac{1756679}{54022501} = \mathcal{H}B_2. \end{aligned}$$

Theorem 4. *(Cassini identity) For $n > 0$, the following identity holds:*

$$\begin{aligned} \mathcal{H}B_{n-1}\mathcal{H}B_{n+1} - (\mathcal{H}B_n)^2 = \Bigg[&\left(\sum_{k=1}^{n} \frac{1}{(M\alpha^k + N\beta^k) + i(M\alpha^{k+1} + N\beta^{k+1})}\right) \\ &- \frac{1}{(M\alpha^{n+1} + N\beta^{n+1}) + i(M\alpha^{n+2} + N\beta^{n+2})} \\ &- \frac{1}{(M\alpha^n + N\beta^n) + i(M\alpha^{n+1} + N\beta^{n+1})} \\ &+ \frac{1}{(M\alpha^n M^* + N\beta^n N^*)(M\alpha^{n+1} M^* + N\beta^{n+1} N^*)} \Bigg] \end{aligned}$$

where $M = \frac{\alpha}{\alpha-\beta}, N = -\frac{\beta}{\alpha-\beta}, M^* = \alpha - i, N^* = \beta - i$.

Proof. From the Binet formula for the harmonic complex balancing numbers;

$$\begin{aligned}
\mathcal{H}B_{n-1}\mathcal{H}B_{n+1}-(\mathcal{H}B_n)^2 &= \left(\sum_{k=1}^{n-1}\frac{1}{(M\alpha^k+N\beta^k)+i(M\alpha^{k+1}+N\beta^{k+1})}\right)\\
&\quad\times\left(\sum_{k=1}^{n+1}\frac{1}{(M\alpha^k+N\beta^k)+i(M\alpha^{k+1}+N\beta^{k+1})}\right)\\
&\quad-\left(\sum_{k=1}^{n}\frac{1}{(M\alpha^k+N\beta^k)+i(M\alpha^{k+1}+N\beta^{k+1})}\right)^2\\
&= \left[\left(\sum_{k=1}^{n}\frac{1}{(M\alpha^k+N\beta^k)+i(M\alpha^{k+1}+N\beta^{k+1})}\right)\right.\\
&\quad-\frac{1}{(M\alpha^{n+1}+N\beta^{n+1})+i(M\alpha^{n+2}+N\beta^{n+2})}\\
&\quad-\frac{1}{(M\alpha^{n}+N\beta^{n})+i(M\alpha^{n+1}+N\beta^{n+1})}\\
&\quad\left.+\frac{1}{(M\alpha^nM^*+N\beta^nN^*)(M\alpha^{n+1}M^*+N\beta^{n+1}N^*)}\right].
\end{aligned}$$

Therefore, the proof is completed. □

Theorem 5. *Let $\mathcal{H}B_n$ be the nth harmonic complex balancing number. Then, we have*

$$\sum_{k=0}^{n-1}\mathcal{H}B_k=n\mathcal{H}B_n-\sum_{k=0}^{n-1}\frac{k+1}{\mathbb{B}_{k+1}}. \tag{21}$$

Proof. Let $u(k)=\mathcal{H}B_k$ and $\Delta v(k)=1$ be as in (9). Then, we obtain $v(k)=k$, $\Delta u(k)=\frac{1}{\mathbb{B}_{k+1}}$, and $Ev(k)=k+1$. Hence, we have

$$\sum_{k=0}^{n-1}\mathcal{H}B_k=n\mathcal{H}B_n-\sum_{k=0}^{n-1}\frac{k+1}{\mathbb{B}_{k+1}}.$$

□

Example 3. *For n=2,*

$$\begin{aligned}
\sum_{k=0}^{1}\mathcal{H}B_k &= 2\mathcal{H}B_2-\sum_{k=0}^{1}\frac{k+1}{\mathbb{B}_{k+1}}\\
&= 2\mathcal{H}B_2-\left(\frac{1}{\mathbb{B}_1}+\frac{2}{\mathbb{B}_2}\right)\\
&= 2\left(\frac{301181-1756679i}{54022501}\right)-\left(\frac{1}{6+35i}+\frac{2}{35+204i}\right)\\
&= \frac{6-35i}{1261}=\mathcal{H}B_0+\mathcal{H}B_1.
\end{aligned}$$

Theorem 6. *Let $\mathcal{H}B_n$ be the nth harmonic complex balancing number. Then, we have*

$$\sum_{k=0}^{n-1}(\mathcal{H}B_k)^2=n(\mathcal{H}B_n)^2-\sum_{k=0}^{n-1}\frac{k+1}{\mathbb{B}_{k+1}}\left(2\mathcal{H}B_k+\frac{1}{\mathbb{B}_{k+1}}\right). \tag{22}$$

Proof. Let $u(k) = \mathcal{H}B_k^2$ and $\Delta v(k) = 1$ be as in (9). Then, by using (9), we obtain $v(k) = k$, $\Delta u(k) = \frac{1}{\mathbb{B}_{k+1}}\left(2\mathcal{H}B_k + \frac{1}{\mathbb{B}_{k+1}}\right)$ and $Ev(k) = k+1$. Hence, we have

$$\sum_{k=0}^{n-1}(\mathcal{H}B_k)^2 = n(\mathcal{H}B_n)^2 - \sum_{k=0}^{n-1}\frac{k+1}{\mathbb{B}_{k+1}}\left(2\mathcal{H}B_k + \frac{1}{\mathbb{B}_{k+1}}\right).$$

□

Example 4. *For n = 2,*

$$\begin{aligned}
\sum_{k=0}^{1}(\mathcal{H}B_k)^2 &= 2(\mathcal{H}B_2)^2 - \sum_{k=0}^{1}\frac{k+1}{\mathbb{B}_{k+1}}\left(2\mathcal{H}B_k + \frac{1}{\mathbb{B}_{k+1}}\right)\\
&= 2(\mathcal{H}B_2)^2 - \left(\frac{1}{\mathbb{B}_1}\left(2\mathcal{H}B_0 + \frac{1}{\mathbb{B}_1}\right) + \frac{2}{\mathbb{B}_2}\left(2\mathcal{H}B_1 + \frac{1}{\mathbb{B}_2}\right)\right)\\
&= 2\left(\frac{301181 - 1756679i}{54022501}\right)^2 - \left(\frac{1}{(6+35i)^2} + \frac{2}{(35+204i)}\left(\frac{558227 - 3256114i}{54022501}\right)\right)\\
&= \frac{-5990422228560 - 2116313351596i}{2918430614295001} + \frac{3808189555451 + 1345465813576i}{2918430614295001}\\
&= \frac{-1189 - 420i}{1590121} = (\mathcal{H}B_0)^2 + (\mathcal{H}B_1)^2.
\end{aligned}$$

Theorem 7. *Let $\mathcal{H}B_n$ be the nth harmonic complex balancing number and m be a non-negative integer; then,*

$$\sum_{k=0}^{n-1}\binom{k}{m}(\mathcal{H}B_k) = \binom{n}{m+1}\mathcal{H}B_n - \sum_{k=0}^{n-1}\binom{k+1}{m+1}\frac{1}{\mathbb{B}_{k+1}}. \tag{23}$$

Proof. Let $u(k) = \mathcal{H}B_k$ and $\Delta v(k) = \binom{k}{m}$ be as in (9). Then, we obtain $v(k) = \binom{k}{m+1}$, $\Delta u(k) = \frac{1}{\mathbb{B}_{k+1}}$ and $Ev(k) = \binom{k+1}{m+1}$. Hence, we have

$$\sum_{k=0}^{n-1}\binom{k}{m}(\mathcal{H}B_k) = \binom{n}{m+1}\mathcal{H}B_n - \sum_{k=0}^{n-1}\binom{k+1}{m+1}\frac{1}{\mathbb{B}_{k+1}}.$$

□

Example 5. *For n = 3 and m = 0,*

$$\begin{aligned}
\sum_{k=0}^{2}\binom{k}{m}(\mathcal{H}B_k) &= \binom{n}{m+1}\mathcal{H}B_n - \sum_{k=0}^{n-1}\binom{k+1}{m+1}\frac{1}{\mathbb{B}_{k+1}}\\
&= \binom{3}{1}\mathcal{H}B_3 - \sum_{k=0}^{2}\binom{k+1}{1}\frac{1}{\mathbb{B}_{k+1}} = 3\mathcal{H}B_3 - \left(\frac{1}{\mathbb{B}_1} + \frac{2}{\mathbb{B}_2} + \frac{3}{\mathbb{B}_3}\right)\\
&= \frac{558227 - 3256114i}{54022501} = \binom{1}{0}\mathcal{H}B_1 + \binom{2}{0}\mathcal{H}B_2.
\end{aligned}$$

3. On Symmetric Matrices and the Harmonic Complex Balancing Numbers

In this section, we consider a particular $n \times n$ matrix $\mathcal{HB} = [\mathcal{HB}_{k_{ij}}]_{i,j=1}^{n}$, where $k_{i,j} = min(i,j)$ and $\mathcal{HB}_n$ is the n^{th} harmonic complex balancing number. In other words, these matrices are represented as follows:

$$\mathcal{HB} = [\mathcal{HB}_{min(i,j)}]_{i,j=1}^{n} = \begin{bmatrix} \mathcal{HB}_1 & \mathcal{HB}_1 & \mathcal{HB}_1 & \cdots & \mathcal{HB}_1 \\ \mathcal{HB}_1 & \mathcal{HB}_2 & \mathcal{HB}_2 & \cdots & \mathcal{HB}_2 \\ \mathcal{HB}_1 & \mathcal{HB}_2 & \mathcal{HB}_3 & \cdots & \mathcal{HB}_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{HB}_1 & \mathcal{HB}_2 & \mathcal{HB}_3 & \cdots & \mathcal{HB}_n \end{bmatrix} \tag{24}$$

$$= \begin{bmatrix} \frac{6-35i}{1261} & \frac{6-35i}{1261} & \cdots & \frac{6-35i}{1261} \\ \frac{6-35i}{1261} & \frac{301181-1756679i}{54022501} & \cdots & \frac{301181-1756679i}{54022501} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{6-35i}{1261} & \frac{301181-1756679i}{54022501} & \cdots & \sum_{k=1}^{n} \frac{1}{\mathbb{B}_n} \end{bmatrix}.$$

We investigate some linear algebraic properties for them. Then, we compute some norms for these matrices and obtain some interesting inequalities.

Theorem 8. *Let $\mathcal{HB}$ be an n-square matrix as in (24); then,*

$$det(\mathcal{HB}) = \mathcal{HB}_1 \prod_{i=2}^{n} (\mathcal{HB}_i - \mathcal{HB}_{i-1}) = \frac{1}{\mathbb{B}_1 \mathbb{B}_2 \cdots \mathbb{B}_n}$$

where $\mathbb{B}_n$ is the nth complex balancing number.

Proof. Using elementary row operations on the matrix (24), we have

$$det(\mathcal{HB}) = det \begin{bmatrix} \mathcal{HB}_1 & \mathcal{HB}_1 & \mathcal{HB}_1 & \cdots & \mathcal{HB}_1 \\ 0 & \mathcal{HB}_2 - \mathcal{HB}_1 & \mathcal{HB}_2 - \mathcal{HB}_1 & \cdots & \mathcal{HB}_2 - \mathcal{HB}_1 \\ 0 & 0 & \mathcal{HB}_3 - \mathcal{HB}_2 & \cdots & \mathcal{HB}_3 - \mathcal{HB}_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mathcal{HB}_n - \mathcal{HB}_{n-1} \end{bmatrix}.$$

Therefore, we have

$$det(\mathcal{HB}) = \mathcal{HB}_1 \prod_{i=2}^{n} (\mathcal{HB}_i - \mathcal{HB}_{i-1}) = \frac{1}{\mathbb{B}_1 \mathbb{B}_2 \cdots \mathbb{B}_n}.$$

□

Theorem 9. *Let $\mathcal{HB}$ be a matrix as in (24); then, $\mathcal{HB}$ is invertible and the inverse of $\mathcal{HB}$ is a symmetric tridiagonal matrix of the form*

$$\mathcal{HB}^{-1} = \begin{bmatrix} \mathbb{B}_1 + \mathbb{B}_2 & -\mathbb{B}_2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -\mathbb{B}_2 & \mathbb{B}_2 + \mathbb{B}_3 & -\mathbb{B}_3 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\mathbb{B}_3 & \mathbb{B}_3 + \mathbb{B}_4 & -\mathbb{B}_4 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -\mathbb{B}_4 & \mathbb{B}_4 + \mathbb{B}_5 & -\mathbb{B}_5 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -\mathbb{B}_{n-1} & \mathbb{B}_{n-1} + \mathbb{B}_n & -\mathbb{B}_n \\ 0 & 0 & 0 & \cdots & 0 & 0 & -\mathbb{B}_n & \mathbb{B}_n \end{bmatrix}.$$

Proof. By Theorem 8, it is known that $\mathcal{HB}$ is nonsingular, so it is invertible.

Let us prove the inverse using principle mathematical induction (PMI) on n. It verifies for $n = 2$, i.e.,:

$$\mathcal{HB}_{2\times 2} = \begin{bmatrix} \frac{6-35i}{1261} & \frac{6-35i}{1261} \\ \frac{6-35i}{1261} & \frac{301181-1756679i}{54022501} \end{bmatrix}.$$

Then, we have

$$\mathcal{HB}_{2\times 2}^{-1} = \begin{bmatrix} 41+239i & -35-204i \\ -35-204i & 35+204i \end{bmatrix}.$$

Thus, by taking $b = (\mathcal{HB}_1, \mathcal{HB}_2, \cdots, \mathcal{HB}_n)^T, b^T = (\mathcal{HB}_1, \mathcal{HB}_2, \cdots, \mathcal{HB}_n)$ and $c = \mathcal{HB}_{n+1}$ along with Equation (10), the proof is completed for $n+1$. Therefore, the result is true for each n. □

Example 6. *For $n = 3$,*

$$\mathcal{HB} = [\mathcal{HB}_{min(i,j)}]_{i,j=1}^{3} = \begin{bmatrix} \mathcal{HB}_1 & \mathcal{HB}_1 & \mathcal{HB}_1 \\ \mathcal{HB}_1 & \mathcal{HB}_2 & \mathcal{HB}_2 \\ \mathcal{HB}_1 & \mathcal{HB}_2 & \mathcal{HB}_3 \end{bmatrix}$$

$$\mathcal{HB}_{3\times 3} = \begin{bmatrix} \frac{6-35i}{1261} & \frac{6-35i}{1261} & \frac{6-35i}{1261} \\ \frac{6-35i}{1261} & \frac{301181-1756679i}{54022501} & \frac{301181-1756679i}{54022501} \\ \frac{6-35i}{1261} & \frac{301181-1756679i}{54022501} & \frac{34564649477-201599438424i}{6047764964449} \end{bmatrix},$$

where $\mathcal{HB}_{3\times 3}$ is the third harmonic complex balancing number. The inverse of the$\mathcal{HB}_{3\times 3}$ matrix is as follows:

$$\mathcal{HB}_{3\times 3}^{-1} = \begin{bmatrix} 41+239i & -35-204i & 0 \\ -35-204i & 239+1393i & -204-1189i \\ 0 & -204-1189i & 204+1189i \end{bmatrix}.$$

Theorem 10. *Let $\mathcal{HB}$ be a matrix as in (24); then, the Euclidean norm of $\mathcal{HB}$ is*

$$\|\mathcal{HB}\|_{\mathbb{E}} = \sqrt{(n+1)\left[(n+1)\mathcal{HB}_{n+1}^2 - (3n+4)\mathcal{HB}_{n+1} + \frac{7n+6}{2}\right]}.$$

Proof. From the definition of the Euclidean norm, we have

$$\|\mathcal{HB}\|_{\mathbb{E}}^2 = \left[\left(\sum_{i=1}^{n}\sum_{j=1}^{n} |\mathcal{HB}_{k_{ij}}|^2\right)^{\frac{1}{2}}\right]^2.$$

Thus,

$$\begin{aligned} \|\mathcal{HB}\|_{\mathbb{E}}^2 &= \sum_{k=1}^{n}(2n-2k+1)(\mathcal{HB}_k)^2 = (2n+1)\sum_{k=1}^{n}\mathcal{HB}_k^2 - 2\sum_{k=1}^{n}k\mathcal{HB}_k^2 \\ &= (2n+1)\sum_{k=1}^{n}\mathcal{HB}_k^2 - 2\left(\sum_{k=1}^{n}k\mathcal{HB}_{k+1}^2 + \sum_{k=1}^{n}\mathcal{HB}_k^2 - n\mathcal{HB}_{n+1}^2\right) \\ &= (2n-1)\sum_{k=1}^{n}\mathcal{HB}_k^2 - 2\sum_{k=1}^{n}k\mathcal{HB}_{k+1}^2 + 2n\mathcal{HB}_{n+1}^2. \end{aligned}$$

According to (6) and (7), we obtain

$$\begin{aligned}\|\mathcal{HB}\|_{\mathbb{E}}^2 &= (2n-1)\Big[(n+1)\mathcal{HB}_{n+1}^2-(2n+3)\mathcal{HB}_{n+1}+(2n+2)\Big]+2n\mathcal{HB}_{n+1}^2\\ &\quad -2\left[\frac{(n+2)(n-1)}{2}\mathcal{HB}_{n+1}^2-\frac{n^2-3n-7}{2}\mathcal{HB}_{n+1}+\frac{n^2-9n-10}{4}\right].\end{aligned}$$

By some computations, we have

$$\begin{aligned}\|\mathcal{HB}\|_{\mathbb{E}}^2 &= (n+1)^2\mathcal{HB}_{n+1}^2-(3n^2+7n+4)\mathcal{HB}_{n+1}+\frac{7n^2+13n+6}{2}\\ &= (n+1)^2\mathcal{HB}_{n+1}^2-(3n+4)(n+1)\mathcal{HB}_{n+1}+\frac{(7n+6)(n+1)}{2}.\end{aligned}$$

Thus, by taking the square root of the equation,

$$\|\mathcal{HB}\|_{\mathbb{E}}=\sqrt{(n+1)\left[(n+1)\mathcal{HB}_{n+1}^2-(3n+4)\mathcal{HB}_{n+1}+\frac{7n+6}{2}\right]}.$$

Thus, the proof is completed. □

Corollary 1. *Let $\mathcal{HB}$ be a matrix as in (24); then, we have the following upper and lower bounds for the spectral norm of $\mathcal{HB}$.*

$$\frac{1}{\sqrt{n}}\sqrt{(n+1)\left[(n+1)\mathcal{HB}_{n+1}^2-(3n+4)\mathcal{HB}_{n+1}+\frac{7n+6}{2}\right]}\le\|\mathcal{HB}\|_2$$

$$\le\sqrt{(n+1)\left[(n+1)\mathcal{HB}_{n+1}^2-(3n+4)\mathcal{HB}_{n+1}+\frac{7n+6}{2}\right]}.$$

Theorem 11. *Let $\mathcal{HB}$ be a matrix as in (24); then, we have the following upper bound for the spectral norm of $\mathcal{HB}$, as follows:*

$$\|\mathcal{HB}\|_2\le\sqrt{((n+1)\mathcal{HB}_{n+1}^2-(2n+3)\mathcal{HB}_{n+1}+2n+2)(n\mathcal{HB}_n^2-(2n+1)\mathcal{HB}_n+2n+1)}.$$

Proof. By the definition of the Hadamard product for matrix $\mathcal{HB}$, we have

$$\mathcal{HB}=V\circ P$$

where

$$V=\begin{bmatrix}\mathcal{HB}_1 & 1 & 1 & \cdots & 1\\ \mathcal{HB}_1 & \mathcal{HB}_2 & 1 & \cdots & 1\\ \mathcal{HB}_1 & \mathcal{HB}_2 & \mathcal{HB}_3 & \cdots & 1\\ \vdots & \vdots & \vdots & \vdots & \\ \mathcal{HB}_1 & \mathcal{HB}_2 & \mathcal{HB}_3 & \cdots & \mathcal{HB}_n\end{bmatrix}$$

and

$$P=\begin{bmatrix}1 & \mathcal{HB}_1 & \mathcal{HB}_1 & \cdots & \mathcal{HB}_1\\ 1 & 1 & \mathcal{HB}_2 & \cdots & \mathcal{HB}_2\\ 1 & 1 & 1 & \cdots & \mathcal{HB}_3\\ \vdots & \vdots & \vdots & \vdots & \\ 1 & 1 & 1 & \cdots & 1\end{bmatrix}.$$

By the definition of maximum row length norm and maximum column length norm, we have

$$r_1(V) = max_i \sqrt{\sum_j |a_{ij}|^2}$$

$$= \sqrt{\sum_{i=1}^{n} \mathcal{H}B_i^2} = \sqrt{(n+1)\mathcal{H}B_{n+1}^2 - (2n+3)\mathcal{H}B_{n+1} + 2n + 2},$$

and

$$c_1(P) = max_j \sqrt{\sum_i |b_{ij}|^2}$$

$$= \sqrt{\sum_{i=1}^{n-1} \mathcal{H}B_i^2 + 1} = \sqrt{n\mathcal{H}B_n^2 - (2n+1)\mathcal{H}B_n + 2n + 1}.$$

Consequently, by the Hadamard product (14), we obtain

$$\|\mathcal{H}B\|_2 \leq \sqrt{((n+1)\mathcal{H}B_{n+1}^2 - (2n+3)\mathcal{H}B_{n+1} + 2n + 2)(n\mathcal{H}B_n^2 - (2n+1)\mathcal{H}B_n + 2n + 1)}.$$

□

Theorem 12. *Let $\mathcal{H}B$ be a matrix as in (24); the determinant of the Hadamard inverse is*

$$det(\mathcal{H}B^{\circ(-1)}) = \prod_{k=2}^{n} \frac{-1}{\mathcal{H}B_{k-1}(k\mathcal{H}B_{k-1} + 1)}.$$

Proof. Let $\mathcal{H}B$ be a matrix as in matrix (24). Then, by definition of the Hadamard inverse, we have

$$\mathcal{H}B^{\circ(-1)} = \begin{bmatrix} \frac{1}{\mathcal{H}B_1} & \frac{1}{\mathcal{H}B_1} & \frac{1}{\mathcal{H}B_1} & \cdots & \frac{1}{\mathcal{H}B_1} \\ \frac{1}{\mathcal{H}B_1} & \frac{1}{\mathcal{H}B_2} & \frac{1}{\mathcal{H}B_2} & \cdots & \frac{1}{\mathcal{H}B_2} \\ \frac{1}{\mathcal{H}B_1} & \frac{1}{\mathcal{H}B_2} & \frac{1}{\mathcal{H}B_3} & \cdots & \frac{1}{\mathcal{H}B_3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\mathcal{H}B_1} & \frac{1}{\mathcal{H}B_2} & \frac{1}{\mathcal{H}B_3} & \cdots & \frac{1}{\mathcal{H}B_n} \end{bmatrix}.$$

By using elemantary row operations, we have

$$det(\mathcal{H}B^{\circ(-1)}) = det \begin{bmatrix} \frac{1}{\mathcal{H}B_1} & \frac{1}{\mathcal{H}B_1} & \frac{1}{\mathcal{H}B_1} & \cdots & \frac{1}{\mathcal{H}B_1} \\ 0 & \frac{1}{\mathcal{H}B_2} - \frac{1}{\mathcal{H}B_1} & \frac{1}{\mathcal{H}B_2} - \frac{1}{\mathcal{H}B_1} & \cdots & \frac{1}{\mathcal{H}B_2} - \frac{1}{\mathcal{H}B_1} \\ 0 & 0 & \frac{1}{\mathcal{H}B_3} - \frac{1}{\mathcal{H}B_2} & \cdots & \frac{1}{\mathcal{H}B_3} - \frac{1}{\mathcal{H}B_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\mathcal{H}B_n} - \frac{1}{\mathcal{H}B_{n-1}} \end{bmatrix}.$$

Therefore, we obtain

$$\begin{aligned} det(\mathcal{H}B^{\circ(-1)}) &= \frac{1}{\mathcal{H}B_1} \prod_{k=2}^{n} \left(\frac{1}{\mathcal{H}B_k} - \frac{1}{\mathcal{H}B_{k-1}} \right) = \frac{1}{\mathcal{H}B_1} \prod_{k=2}^{n} \left(\frac{\mathcal{H}B_{k-1} - \mathcal{H}B_k}{\mathcal{H}B_k \mathcal{H}B_{k-1}} \right) \\ &= \frac{1}{\mathcal{H}B_1} \prod_{k=2}^{n} \left(\frac{\frac{-1}{B_k + iB_{k+1}}}{\mathcal{H}B_k \mathcal{H}B_{k-1}} \right) = \frac{1}{\mathcal{H}B_1} \prod_{k=2}^{n} \left(\frac{-1}{\mathbb{B}_k(\mathcal{H}B_{k-1}\mathcal{H}B_k)} \right). \end{aligned}$$

□

4. Numerical Examples

In this section, we give illustrative examples of how we calculate our results.

Example 7. *Let $\mathcal{H}B$ be a matrix as in (24). The determinants of $\mathcal{H}B$ for some values of n are represented in Table 2.*

Table 2. Determinants.

n	Det($\mathcal{H}B$) (for $n > 2$ Is Rounded Off to Four Decimal Places)
2	$-0.00012828 - 0.000045333i$
3	$-5.50182 \times 10^{-8} + 9.84493 \times 10^{-8}i$
4	$1.24768 \times 10^{-11} + 1.00798 \times 10^{-11}i$
5	$2.93903 \times 10^{-16} - 2.58475 \times 10^{-16}i$
6	$-8.58481 \times 10^{-22} - 1.39573 \times 10^{-21}i$
7	$-1.09241 \times 10^{-27} + 4.38239 \times 10^{-28}i$

Example 8. *Let $\mathcal{H}B$ be a matrix as in (24) for $n = 4$. Then, the matrix $\mathcal{H}B_{4\times4}$ is*

$$\mathcal{H}B_{4\times4} =$$

$$\begin{bmatrix} \frac{6-35i}{1261} & \frac{6-35i}{1261} & \frac{6-35i}{1261} & \frac{6-35i}{1261} \\ \frac{6-35i}{1261} & \frac{301181-1756679i}{54022501} & \frac{301181-1756679i}{54022501} & \frac{301181-1756679i}{54022501} \\ \frac{6-35i}{1261} & \frac{301181-1756679i}{54022501} & \frac{34564649477-201599438424i}{6047764964449} & \frac{34564649477-201599438424i}{6047764964449} \\ \frac{6-35i}{1261} & \frac{301181-1756679i}{54022501} & \frac{34564649477-201599438424i}{6047764964449} & \frac{1716019398033981078-10008709241260604874i}{298993159974472584829} \end{bmatrix}.$$

From Theorem 8, the determinant of $\mathcal{H}B_{4\times4}$ can be calculated as

$$\det(\mathcal{H}B_{4\times4}) = \frac{1}{\mathbb{B}_1\mathbb{B}_2\mathbb{B}_3\mathbb{B}_4} = \frac{3730483847 + 3013795632i}{298993159974472584829}.$$

The inverse of the matrix $\mathcal{H}B_{4\times4}$ can be written as

$$\mathcal{H}B^{-1}_{4\times4} = \begin{bmatrix} 41 + 239i & -35 - 204i & 0 & 0 \\ -35 - 204i & 239 + 1393i & -204 - 1189i & 0 \\ 0 & -204 - 1189i & 1393 + 8119i & -1189 - 6930i \\ 0 & 0 & -1189 - 6930i & 1189 + 6930i \end{bmatrix}.$$

By Theorem 10, the Euclidean norm of the matrix $\mathcal{H}B$ can be obtained as

$$||\mathcal{H}B||_E \approx 10.0971 + 0.157112i.$$

From Corollary 1, the spectral norm of $\mathcal{H}B$ can be given between the following inequalities:

$$5.04856 + 0.0785561i \leq ||\mathcal{H}B||_2 \leq 10.0971 + 0.157112i.$$

Let us define the following two matrix

$$V = \begin{bmatrix} \mathcal{H}B_1 & 1 & 1 & 1 \\ \mathcal{H}B_1 & \mathcal{H}B_2 & 1 & 1 \\ \mathcal{H}B_1 & \mathcal{H}B_2 & \mathcal{H}B_3 & 1 \\ \mathcal{H}B_1 & \mathcal{H}B_2 & \mathcal{H}B_3 & \mathcal{H}B_4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{6-35i}{1261} & 1 & 1 & 1 \\ \frac{6-35i}{1261} & \frac{301181-1756679i}{54022501} & 1 & 1 \\ \frac{6-35i}{1261} & \frac{301181-1756679i}{54022501} & \frac{34564649477-201599438424i}{6047764964449} & 1 \\ \frac{6-35i}{1261} & \frac{301181-1756679i}{54022501} & \frac{34564649477-201599438424i}{6047764964449} & \frac{298993159974472584829}{1716019398033981078-10008709241260604874i} \end{bmatrix}$$

and

$$P = \begin{bmatrix} 1 & \mathcal{H}B_1 & \mathcal{H}B_1 & \mathcal{H}B_1 \\ 1 & 1 & \mathcal{H}B_2 & \mathcal{H}B_2 \\ 1 & 1 & 1 & \mathcal{H}B_3 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{6-35i}{1261} & \frac{6-35i}{1261} & \frac{6-35i}{1261} \\ 1 & 1 & \frac{301181-1756679i}{54022501} & \frac{301181-1756679i}{54022501} \\ 1 & 1 & 1 & \frac{34564649477-201599438424i}{6047764964449} \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

The maximum row norm of the matrix V is

$$r_1(V) = 3.15195 + 0.0581486i$$

and the maximum column norm of the matrix P is

$$c_1(P) = 2.99107 + 0.050105i.$$

Then, we have

$$||\mathcal{H}B||_2 \leq 9.42478 + 0.331855i.$$

The Hadamard inverse of $\mathcal{H}B_{4\times4}$ *is*

$$\mathcal{H}B_{4\times4}^{\circ(-1)} = \begin{bmatrix} \frac{1}{\mathcal{H}B_1} & \frac{1}{\mathcal{H}B_1} & \frac{1}{\mathcal{H}B_1} & \frac{1}{\mathcal{H}B_1} \\ \frac{1}{\mathcal{H}B_1} & \frac{1}{\mathcal{H}B_2} & \frac{1}{\mathcal{H}B_2} & \frac{1}{\mathcal{H}B_2} \\ \frac{1}{\mathcal{H}B1} & \frac{1}{\mathcal{H}B_2} & \frac{1}{\mathcal{H}B_3} & \frac{1}{\mathcal{H}B_3} \\ \frac{1}{\mathcal{H}B_1} & \frac{1}{\mathcal{H}B_2} & \frac{1}{\mathcal{H}B_3} & \frac{1}{\mathcal{H}B_4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1261}{6-35i} & \frac{1261}{6-35i} & \frac{1261}{6-35i} & \frac{1261}{6-35i} \\ \frac{1261}{6-35i} & \frac{54022501}{301181-1756679i} & \frac{54022501}{301181-1756679i} & \frac{54022501}{301181-1756679i} \\ \frac{1261}{6-35i} & \frac{54022501}{301181-1756679i} & \frac{6047764964449}{34564649477-201599438424i} & \frac{6047764964449}{34564649477-201599438424i} \\ \frac{1261}{6-35i} & \frac{54022501}{301181-1756679i} & \frac{6047764964449}{34564649477-201599438424i} & \frac{298993159974472584829}{1716019398033981078-10008709241260604874i} \end{bmatrix},$$

respectively. From Theorem 10, the determinant of $\mathcal{H}B_{4\times4}^{\circ(-1)}$ *is*

$$\det(\mathcal{H}B_{4\times4}^{\circ(-1)}) = -13.2201 + 10.6645i.$$

5. Conclusions

In this paper, we consider the balancing numbers, defined by Behera and Panda in [6] and inspire harmonic numbers; then, we define harmonic complex balancing numbers. In this content, we obtain some amazing characteristic properties, such as the Binet formula and Cassini identity, etc. Furthermore, we define a particular $n \times n$ matrix $\mathcal{H}B = [\mathcal{H}B_{k_{i,j}}]_{i,j=1}^{n}$ with harmonic complex balancing entries. The determinants and inverses of these matrices are investigated. Moreover, we give some norms and some bounds for these matrices. Finally, we illustrate the results with some examples.

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